Polar spaces and embeddings of classical groups

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Abstract

Given polar spaces (V,β) and (V,Q) where V is a vector space over a field K, β a reflexive sesquilinear form and Q a quadratic form, we have associated classical isometry groups. Given a subfield F of K and an F-linear function $L : K \to F$ we can define new spaces $(V, L\beta)$ and (V, LQ) which are polar spaces over F.

The construction so described gives an embedding of the isometry groups of (V,β) and (V,Q) into the isometry groups of $(V, L\beta)$ and (V, LQ). In the finite field case under certain added restrictions these subgroups are maximal and form the so called *field extension subgroups* of Aschbacher's class $\mathcal{C}_3$ [1].

We give precise descriptions of the polar spaces so defined and their associated isometry group embeddings. In the finite field case our results give extra detail to the account of maximal field extension subgroups given by Kleidman and Liebeck [3, p112].

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1 Introduction

Let (V,β) and (V,Q) be polar spaces over a field K with $\beta : V \times V \to K$ a reflexive σ-sesquilinear form where σ is a K-automorphism and $Q : V \to K$ a quadratic form with polar form $f_Q : V \times V \to K$. Let F be a subfield of K and $L : K \to F$ an F-linear function. We now compose functions to get $L\beta : V \times V \to F$ and $LQ : V \to F$ regarding V as a vector space over F. In order for these forms to be well-defined it is necessary to impose the condition $\sigma(F) = F$ after which it is easily verified that LQ is a quadratic form with polar form Lf_Q and $L\beta$ is a sesquilinear form. In fact if $F \subseteq Fix(\sigma)$ then β is bilinear.

We present three results on this situation: In Section 2, Theorem A gives conditions on the degeneracy of our composed forms, $L\beta$ and LQ. In Section 3, Theorem B gives conditions on the type (alternating, symmetric or hermitian) of our composed forms. In sections 4 and 5 we consider the situation where our fields are finite. Theorem C summarises these results and gives the isometry group embeddings which are induced by these composed forms.

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2 Results on degeneracy

We begin be presenting results on degeneracy. Our definition of degeneracy is consistent with that of Taylor [5] and so is slightly more general than that of Kleidman and Liebeck [3]:

Definition 1. *A σ-sesquilinear form β is* non-degenerate *if*

$$\beta(u,v) = 0, \forall v \in V \implies u = 0.$$

A quadratic form Q is non-degenerate *if its polar form f_Q has the property that*

$$f_Q(u,v) = Q(u) = 0, \forall v \in V \implies u = 0.$$

The forms are called degenerate *otherwise.*

Our first result concerns sesquilinear forms and uses an adaptation of a proof given by Lam [4]:

Lemma 2. *$L\beta$ is non-degenerate exactly when $L \neq 0$ and β is non-degenerate.*

Proof. If either $L = 0$ or β is degenerate then it is clear that $L\beta$ will be degenerate. Now suppose that $L \neq 0$, β is non-degenerate and $L\beta$ is degenerate. Then there exists nonzero $v \in V$ such that $L\beta(v,w) = 0$ for all $w \in V$. Note that there exists $w \in V$ such that $\beta(v,w) \neq 0$. Now consider, for any $c \in K$,

$$\begin{aligned} \beta(v, \frac{\sigma(c)}{\sigma(\beta(v,w))}w) &= \sigma(\frac{\sigma(c)}{\sigma(\beta(v,w))})\beta(v,w) \\ &= c \end{aligned}$$

Then $L\beta(v, \frac{\sigma(c)}{\sigma(\beta(v,w))}w) = Lc = o$ for all $c \in K$. This implies that $L = 0$ which is a contradiction. □

We turn our attention to quadratic forms. To begin with we can apply the previous lemma directly to get the following:

Lemma 3. *If $L = 0$ or Q is degenerate then LQ is degenerate. If f_Q is non-degenerate then LQ is non-degenerate.*

Thus we are left with the question of what happens when Q is non-degenerate and f_Q is degenerate. This can only occur in characteristic 2. We are able to present results only for the case where V is finite-dimensional and K is finite, in which case we have the following well-known result (see, for example [5, p. 143]):

Theorem 4. *A non-degenerate quadratic form Q on a vector space V over $GF(2^h)$ has a degenerate associated polar form if and only if dim V is odd, in which case the radical of f_Q, $rad(V, f_Q)$, is of dimension 1.*

Corollary 5. *Let K be a finite field of characteristic 2. Suppose $dim_K V$ is odd, f_Q is degenerate and Q is non-degenerate. Then LQ is degenerate.*

Proof. Take $x \in rad(V, f_Q)$. Then $x \in rad(V, Lf_Q)$. Hence $rad(V, Lf_Q) \supseteq rad(V, f_Q)$. But $dim_F(rad(V, Lf_Q)) \geq dim_F(rad(V, f_Q)) > 1$. Hence LQ is degenerate. □

We can summarise our main results in the following:

Theorem A. *Let $\beta : V \times V \to K$ be a reflexive σ-sesqulinear form. Let $Q : V \to K$ be a quadratic form. let F be a subfield of K and $L : K \to F$ be a F-linear function. Then:*

- *$L\beta$ is non-degenerate if and only if β is non-degenerate and $L \neq 0$;*
- *If char $K \neq 2$, or $K = GF(2^h)$ for some integer h and $dim_K V$ is even, then LQ is non-degenerate if and only if Q is non-degenerate and $L \neq 0$;*
- *If $K = GF(2^h)$ for some integer h and $dim_K V$ is odd then LQ is degenerate;*

Unsolved. *We have failed to ascertain the conditions under which LQ is degenerate in the case where char $K = 2$, $|K| + dim_K V$ is infinite, Q is non-degenerate and f_Q is degenerate.*

3 A classification of β into form

Taking reflexive sesquilinear form $\beta : V \times V \to K$ to be alternating, symmetric or hermition, $L : K \to F$, F-linear and not identically zero, we seek to classify $L\beta$ into these three categories or else as being 'atypical', i.e. not of of this form.

The conditions under which β is hermitian, *char* $K = 2$ and $L\beta$ is alternating will prove to be the most difficult and we discuss this case first. Observe that we must have $F \subseteq Fix(\sigma)$.

Let σ be the field automorphism of order 2 associated with β. It is easily shown that $K/Fix(\sigma)$ is a Galois extension and we may therefore define a trace function:

$$Tr_{K/Fix(\sigma)} : K \to Fix(\sigma), x \mapsto x + \sigma(x).$$

Now any $Fix(\sigma)$-function $L : K \to Fix(\sigma)$ can be written in the form, for some $\alpha \in K$,

$$L : K \to Fix(\sigma), x \mapsto Tr_{K/Fix(\sigma)}(\alpha x).$$

Lemma 6. *When char $K = 2$ and β is hermitian, $L\beta$ is alternating if and only if $F \subseteq Fix(\sigma)$ and $L\sigma = L$.*

Proof. Write $L : K \to F, x \mapsto L_1 \circ Tr_{K/Fix(\sigma)}(\alpha x)$ for some $\alpha \in K$ and some $L_1 : Fix(\sigma) \to F$, F-linear and not identically zero. We suppose that $Tr_{K/Fix(\sigma)}(\alpha\sigma) = Tr_{K/Fix(\sigma)}(\alpha)$ and it is enough to prove that $Tr_{K/Fix(\sigma)}(\alpha\beta)$ is alternating. Now for $x \in K$,

$$\begin{aligned} Tr_{K/Fix(\sigma)}(\alpha\sigma(x)) = Tr_{K/Fix(\sigma)}(\alpha x) \quad &\Longrightarrow \quad \alpha\sigma(x) + \sigma(\alpha\sigma(x)) = \alpha x + \sigma(\alpha x) \\ &\Longrightarrow \quad \alpha\sigma(x) + \sigma(\alpha)x = \alpha x + \sigma(\alpha)\sigma(x) \\ &\Longrightarrow \quad (\sigma(\alpha) + \alpha)(\sigma(x) + x) = 0. \end{aligned}$$

Since $\sigma(x)+x\neq 0$ for all $x\notin Fix(\sigma)$, we must have $\sigma(\alpha)=\alpha$. Then

$$\begin{aligned} Tr_{K/Fix(\sigma)}(\alpha\beta)(x,x) &= \alpha\beta(x,x)+\sigma(\alpha\beta(x,x)) \\ &= \alpha\beta(x,x)+\sigma(\alpha)\sigma\beta(x,x) \\ &= (\alpha+\sigma(\alpha))\beta(x,x)=0. \end{aligned}$$

□

We are now able to state our main result:

Theorem B. *Let $\beta : V\times V\to K$ be a reflexive sesquilinear form. Let K/F be a field extension with $L : K\to F$ a F-linear function which is not identically zero. Then we classify β into type as follows:*

- *If β is alternating then $L\beta$ is alternating;*
- *If β is symmetric then $L\beta$ is symmetric;*
- *If char $K=2$, K is finite and β is symmetric not alternating then $L\beta$ is symmetric not alternating;*
- *If β is hermitian and $F\not\subseteq Fix(\sigma)$ then*
  1. *$L\beta$ is hermitian if and only if $L\sigma=\sigma L$;*
  2. *$L\beta$ is atypical if and only if $L\sigma\neq\sigma L$;*
- *If β is hermitian and $F\subseteq Fix(\sigma)$ then*
  1. *$L\beta$ is symmetric if and only if $L\sigma=L$;*
  2. *$L\beta$ is alternating if and only if char $K\neq 2$ and $L\sigma=-L$ OR char $K=2$ and $L\sigma=L$;*
  3. *$L\beta$ is atypical if and only if $L\sigma\neq\pm L$.*

Proof. The first two statements are self-evident.

We turn to the third statement. Given β symmetric not alternating, $L\beta$ will be alternating if and only if $\{\beta(x,x)|x\in V\}\subseteq null(L)$. Since $L\neq 0$ it is enough to show that $f : V\to K, x\to\beta(x,x)$ is onto. Take any $x\in V$ such that $\beta(x,x)=a\in K^*$. Take any $c\in K$. Then $\beta(\sqrt{\frac{c}{a}}x,\sqrt{\frac{c}{a}}x)=c$ as required.

For the remainder we assume that β is hermitian. First of all suppose that $F\not\subseteq Fix(\sigma)$ so $L\beta$ is σ-sesquilinear. Then $L\beta(v_1,v_2)=L\sigma\beta(v_2,v_1)$ for any $v_1,v_2\in V$ and so $L\beta$ is hermitian if and only if $L\sigma|_{\Im(\beta)}=\sigma L|_{\Im(\beta)}$. Since β is surjective we are done.

Next suppose that $F\subseteq Fix(\sigma)$ in which case $L\beta(v_1,v_2)=L\sigma\beta(v_2,v_1)$. This is symmetric if and only if $L\sigma|_{\Im(\beta)}=L|_{\Im(\beta)}$ and so $L\beta$ is symmetric exactly when $L\sigma=L$.

Now we examine when $L\beta$ is alternating. When *char* K is odd this is equivalent to $L\beta$ being skew-symmetric which, by an analagous argument to the symmetric case, occurs exactly when $L\sigma=-L$. When *char* $K=2$ the previous lemma gives us the required result. The only other possibility is for $L\beta$ to be atypical hence we have our final equivalence. □

Unsolved. *We have failed to ascertain the conditions under which $L\beta$ is alternating in the case where char $K=2$, K is infinite and β is symmetric not alternating.*

4 The isometry classes of (V, LQ) over finite fields

Define $Q : V \to GF(q^w)$ a non-degenerate quadratic form, $L : GF(q^w) \to GF(q)$ a $GF(q)$-linear function which is not the zero function and $Tr_{GF(q^w)/GF(q)}$ the trace function. We restrict V to be a finite A-dimensional vector space over $GF(q^w)$. In order to classify (V, Q) into isometry classes we need to examine the situation when Aw is even and distinguish between the O^+ and O^- cases.

Our first lemma will be useful in distinguishing the isometry class of LQ as well as giving an application of the classification:

Lemma 7. *The isometry group for Q, $Isom(Q, V)$, is a subgroup of the isometry group for LQ, $Isom(LQ, V)$.*

Proof. Simply observe that if $T : V \to V$ satisfies $Q(Tu) = Q(u)$ for all $u \in V$ then $LQ(Tu) = LQ(u)$ for all u in V. $\square$

Consider first the situation when A is even:

Lemma 8. *Let (V, q) have isometry class $O^+(A, q^w)$. Then (V, LQ) has isometry class $O^+(Aw, q)$. Thus $O^+(A, q^w) \leq O^+(Aw, q)$.*

Proof. Let W be a maximal totally singular subspace of (V, q). Then $dim_K W = \frac{1}{2} dim_K V$. But W is also a totally singular subspace of (V, LQ) and $dim_F W = \frac{1}{2} dim_F V$. Thus (V, LQ) is of type $O^+(Aw, q)$. $\square$

Lemma 9. *Let (V, q) have isometry class $O^-(A, q^w)$. Then (V, LQ) has isometry class $O^-(Aw, q)$. Thus $O^-(A, q^w) \leq O^-(Aw, q)$.*

Proof. **Suppose first of all that** $A = 2$. Suppose in addition that (V, LQ) has isometry class $O^+(2w, q)$. Then $O^+(2, q^w) \leq O^-(Aw, q)$ and so, by the theorem of Lagrange,

$$(q^w + 1) \big| q^{w(w-1)} \prod_{i=1}^{w-1} (q^{2i} - 1).$$

If a primitive prime divisor of $q^{2w} - 1$ exists then this is impossible hence we must deal with the exceptions given by Zsigmondy. The first possibility is that $w = 1$, in which case $L : GF(q^w) \to GF(q^w)$ has form $x \mapsto ax$ for some $a \in GF(q^w)^*$. Clearly an element of V is singular under Q exactly when it is singular under LQ. Then (V, q) and (V, LQ) have the same Witt index and hence share type which is a contradiction.

The second possibility is that $(q, w) = (2, 3)$ in which case we must consider whether or not $O^-(2, 8) \leq O^+(6, 2)$. Examining the atlas [2] we see that $O^-(2, 8)$ contains elements of order 9 while $O^+(6, 2)$ does not, hence this possibility can be excluded.

Now suppose that $A = 2m+2$ **for some** $m \geq 1$**.** Then $V = U \perp W$ under Q where U is a direct sum of m hyperbolic lines and W is an anisotropic subspace of dimension 2. Then $Q|_U$ is of type O^+ and hence $LQ|_U$ is of type O^+. Similarly $Q|_W$ is of type O^- and, since $dim_{GF(q^w)} W = 2$, $LQ|_W$ is of type O^-. Then $V = U \perp W$ under LQ, U is a direct sum of mw hyperbolic lines under LQ and W is a direct sum of $w - 1$ hyperbolic lines with a 2-dimensional anisotropic subspace under LQ. Hence (V, LQ) is of type O^-. $\square$

We now consider the situation when A is odd. If the characteristic equals 2 then Theorem A implies that LQ is degenerate so we exclude this situation. We will be interested in the situation where w is even and the characteristic is odd. We will write $L : GF(q^w) \to GF(q)$ in the form, for some $\alpha \in GF(q^w)^*$,

$$L = Tr_{GF(q^w)/GF(q)}(\alpha) : GF(q^w) \to GF(q), x \mapsto \sum_{i=0}^{w-1} (\alpha x)^{q^i}.$$

We will need to work with the *discriminant* of our form LQ for which we will need two preliminary results:

Lemma 10. *Let q be odd, $k \in GF(q^2)\backslash GF(q)$ such that $k^2 \in GF(q)$. Then*

$$Tr_{GF(q^2)/GF(q)}(k) = 0.$$

Proof. Observe that $GF(q^2) = GF(q)(k)$ and k has minimum polynomial $x^2 - k^2$. Now $Gal(GF(q^2)/GF(q))$ acts on the set of roots of this minimum polynomial. Since the trace map is the sum of the elements of $Gal(GF(q^2)/GF(q))$, $Tr_{GF(q^2)/GF(q)}(k) = k - k = 0$. $\square$

Theorem 11. *A non-degenerate quadratic form, $Q : V \to GF(q)$ where V is a $2n$-dimensional vector space over $GF(q)$ and q is odd, gives rise to an $O^+(2n, q)$ space if and only if $disc(Q) \equiv (-1)^n (mod\ GF(q)^{*2})$. Here $GF(q)^{*2}$ is the subgroup of $GF(q)^*$ consisting of all square terms.*

A proof of the previous theorem can be found, for instance, in [3, p.32]. We can now proceed with our study of the type of LQ.

Lemma 12. *Let (V, Q) be of type O and be one-dimensional over $GF(q^2)$, q odd. Then Q has form $Q(u) = \gamma u^2$, for some $\gamma \in GF(q^w)^*$. Then LQ has type,*

$$\begin{aligned} O^+ &\iff (\alpha\gamma)^{-2} \in GF(q)\backslash GF(q)^{*2} \text{ or } (\alpha\gamma)^{q+1} \not\equiv -1 (mod\ GF(q)^{*2}), \\ O^- &\iff (\alpha\gamma)^{-2} \notin GF(q)\backslash GF(q)^{*2} \text{ and } (\alpha\gamma)^{q+1} \equiv -1 (mod\ GF(q)^{*2}). \end{aligned}$$

Proof. Observe that $LQ : V \to GF(q), u \mapsto \alpha\gamma u^2 + (\alpha\gamma u^2)^q$ has polar form $Lf_Q : V \times V \to GF(q), (u, v) \mapsto u^T M v$ where, over a basis for V over $GF(q)$, $\{1, \omega\}$,

$$M = \begin{pmatrix} 2Tr_{GF(q^2)/GF(q)}(\alpha\gamma) & 2Tr_{GF(q^2)/GF(q)}(\alpha\gamma\omega) \\ 2Tr_{GF(q^2)/GF(q)}(\alpha\gamma\omega) & 2Tr_{GF(q^2)/GF(q)}(\alpha\gamma\omega^2) \end{pmatrix}.$$

Now take f to be an element of $GF(q)$ such that $\sqrt{f} \notin GF(q)$. Then $(\alpha\gamma)^{-2} \in GF(q)\backslash GF(q)^{*2}$ if and only if $(\alpha\gamma)^{-1}\sqrt{f} \in GF(q)$.

Suppose that $(\alpha\gamma)^{-1}\sqrt{f} \notin GF(q)$. Let $\omega = (\alpha\gamma)^{-1}\sqrt{f}$. Then

$$M = \begin{pmatrix} 2Tr_{GF(q^2)/GF(q)}(\alpha\gamma) & 2Tr_{GF(q^2)/GF(q)}(\sqrt{f}) \\ 2Tr_{GF(q^2)/GF(q)}(\sqrt{f}) & 2Tr_{GF(q^2)/GF(q)}(\alpha^{-1}\gamma^{-1} f) \end{pmatrix}.$$

Then the discriminant of the form LQ is

$$\begin{aligned} & 4Tr_{GF(q^2)/GF(q)}(\alpha\gamma) Tr_{GF(q^2)/GF(q)}(\alpha^{-1}\gamma^{-1} f) - 4(Tr_{GF(q^2)/GF(q)}(\sqrt{f}))^2 \\ = \ & 4f(\alpha\gamma + (\alpha\gamma)^q)((\alpha\gamma)^{-1} + (\alpha\gamma)^{-q}) \qquad (\text{since } Tr_{GF(q^2)/GF(q)}(\sqrt{f}) = 0) \\ = \ & \frac{f 2^2 (Tr_{GF(q^2)/GF(q)}(\alpha\gamma))^2}{(\alpha\gamma)^{q+1}}. \end{aligned}$$

Referring to Theorem 11 we see that our result holds in this case.

Suppose that $(\alpha\gamma)^{-1}\sqrt{f} \in GF(q)$. Let $\omega = \sqrt{f}$. Then $\alpha\gamma = f_2\sqrt{f}$ for some $f_2 \in GF(q)$ and

$$M = \begin{pmatrix} 2Tr_{GF(q^2)/GF(q)}(f_2\sqrt{f}) & 2Tr_{GF(q^2)/GF(q)}(f_2 f) \\ 2Tr_{GF(q^2)/GF(q)}(f_2 f) & 2Tr_{GF(q^2)/GF(q)}(f_2 f\sqrt{f}) \end{pmatrix}.$$

The discriminant of the form LQ is

$$\begin{aligned} & 4Tr_{GF(q^2)/GF(q)}(f_2\sqrt{f})Tr_{GF(q^2)/GF(q)}(f_2 f\sqrt{f}) - 4(Tr_{GF(q^2)/GF(q)}(f_2 f))^2 \\ = \; & -4(Tr_{GF(q^2)/GF(q)}(f_2 f))^2 \qquad (\text{since } Tr_{GF(q^2)/GF(q)}(\sqrt{f}) = 0). \end{aligned}$$

Appealing to Theorem 11 we conclude that LQ is of isometry class O^+ in all cases here.

$\square$

Lemma 13. *Let (V, Q) be A-dimensional of type O over field $GF(q^w)$ of odd characteristic. Let S be a non-dimensional anisotropic subspace (or germ) where $Q\big|_S$ has form $Q(s) = \gamma s^2$ for some $\gamma \in GF(q^w)^*$. Let $w = 2n$. Then LQ has type*

$$\begin{aligned} O^+ &\iff (\alpha\gamma)^{-2} \in GF(q^n)\backslash GF(q^n)^{*2} \text{ or } (\alpha\gamma)^{q+1} \not\equiv -1 (mod\ GF(q^n)^{*2}), \\ O^- &\iff (\alpha\gamma)^{-2} \notin GF(q^n)\backslash GF(q^n)^{*2} \text{ and } (\alpha\gamma)^{q+1} \equiv -1 (mod\ GF(q^n)^{*2}). \end{aligned}$$

Proof. First take A odd and $w = 2$. Then $(V, Q) = (R, Q\big|_S) \perp (S, Q\big|_R)$ where R is an orthogonal direct sum of orthogonal hyperbolic lines and S is a one-dimensional anisotropic orthogonal space. Then $LQ\big|_R$ is of type O^+ and $LQ\big|_S$ will be either of type O^+ or O^- according to the conditions of the previous lemma. Since $(V, LQ) = (R, LQ\big|_R) \perp (S, LQ\big|_S)$ the type of LQ is determined according to the conditions given.

Now take A odd, w any even number. Then $L = Tr_{GF(q^n)/GF(q)} \circ Tr_{GF(q^w)/GF(q^n)} \circ K$ where $K : GF(q^w) \to GF(q^w), x \mapsto \alpha x$. By the previous paragraph the conditions of the theorem are the conditions under which $Tr_{GF(q^w)/GF(q^n)} \circ K \circ Q$ will be of type O^+ or O^-. By Lemmas 9 and 8 we know that further compositions with $Tr_{GF(q^n)/GF(q)}$ will not change this type. The result follows.

$\square$

We will summarise the results of this section and the next in Theorem C at the end of the paper.

5 The isometry classes of $(V, L\beta)$ over finite fields

Define $\beta : V \times V \to GF(q^w)$ to be a non-degenerate reflexive sesquilinear form of one of the three types, V A-dimensional over $GF(q^w)$. Define $L : GF(q^w) \to GF(q)$, $GF(q)$-linear and not the zero function.

If we consider β symmetric over a field of odd characteristic then β shares isometry class with the quadratic form $Q(v) = \frac{1}{2}\beta(v, v)$ and the results of the previous section give the type of $L\beta$.

Similarly if β is alternating or if the characteristic is 2 and β is symmetric not alternating, then Theorem B gives the type of $L\beta$. Note that over finite fields, symmetric not alternating forms result in polar spaces which are called *pseudo-symplectic*.

In this section we need to consider the case where β is hermitian with automorphism σ. Once again we will take $L = Tr_{GF(q^w)/GF(q)}(\alpha)$ for some α in $GF(q)^*$. Consider first the case where $GF(q) \nsubseteq Fix(\sigma)$ which occurs exactly when w is odd:

Lemma 14. *Let β be hermitian. When w is odd,*

1. *$L\beta$ is hermitian $\iff$ $\sigma(\alpha) = \alpha$;*
2. *$L\beta$ is atypical $\iff$ $\sigma(\alpha) \neq \alpha$.*

When w is even,

1. *$L\beta$ is symmetric $\iff$ $\sigma(\alpha) = \alpha$;*
2. *$L\beta$ is alternating $\iff$ $\sigma(\alpha) = -\alpha$;*
3. *$L\beta$ is atypical $\iff$ $\sigma(\alpha) \neq \pm\alpha$.*

Proof. Observe that $L\beta$ is bilinear if and only if $F(q) \subseteq Fix(\sigma)$ if and only if w is even.

Suppose first that w is odd; then it is enough to prove the first equivalence. By Theorem B we know that $L\beta$ is hermitian if and only if $L\sigma = \sigma L$. Now

$$L\sigma(x) = \sigma L(x) \iff \sigma Tr_{GF(q^w)/GF(q)}((\sigma(\alpha) - \alpha)x) = 0.$$

The surjectivity of the trace function gives us our result.

Now suppose that w is even. By Theorem B it is enough to prove that $L\sigma = \pm L \iff \sigma(\alpha) = \pm\alpha$. Let the Galois group of the field extension $GF(q^w)/GF(q) = \{\sigma_1, \ldots, \sigma_w\}$. Then

$$\begin{aligned} L\sigma = \pm L \quad &\iff \quad \sum_{i=1}^{w} \sigma_i(\alpha\sigma(x)) = \pm \sum_{i=1}^{w} \sigma_i(\alpha x) \\ &\iff \quad \sum_{i=1}^{w} \sigma_i\sigma(\alpha\sigma(x)) = \pm \sum_{i=1}^{w} \sigma_i(\alpha x) \\ &\iff \quad \sum_{i=1}^{w} \sigma_i((\sigma(\alpha) \mp \alpha)x) = 0. \end{aligned}$$

Once again the surjectivity of the trace function gives us our result. □

To complete the classification we need to ascertain the isometry group of $L\beta$ in the case where it is symmetric.

Lemma 15. *Suppose that β is hermitian, w is even, $\sigma(\alpha) = \alpha$ and V is A-dimensional over $GF(q^w)$. Then the isometry class of $L\beta$ is,*

$$O^+(Aw, q) \iff A \text{ is even}, \ O^-(Aw, q) \iff A \text{ is odd}.$$

Proof. If A is even then, with respect to the hermitian form β, V contains a totally isotropic subspace of dimension $\frac{A}{2}$. This subspace is also totally isotropic with respect to $L\beta$ and over $GF(q)$ has dimension $\frac{Aw}{2}$. Hence $(V, L\beta)$ is of type O^+.

Now take A to be odd. First suppose that $A = 1$ and $w = 2$ so that $GF(q) = Fix(\sigma)$. Then, given a basis for V over $GF(q)$, $\{1, \omega\}$, we have $\beta(x, y) = x\sigma(y)$, $L(x) = \alpha x + \sigma(\alpha x)$ and the matrix of $L\beta$ is

$$\begin{pmatrix} 2Tr_{GF(q^w)/GF(q)}(\alpha) & 2Tr_{GF(q^w)/GF(q)}(\alpha\omega) \\ 2Tr_{GF(q^w)/GF(q)}(\alpha\omega) & 2Tr_{GF(q^w)/GF(q)}(\alpha\omega\sigma(\omega)) \end{pmatrix}.$$

Now put $\omega = \sqrt{f}$ where $GF(q^2) = GF(q)(\sqrt{f})$ and the discriminant of $L\beta$ is $-4f\alpha^2$. Since this is minus a non-square, $L\beta$ is of type O^-.

Now let A be any odd integer, $w = 2$. Then $(V, \beta) = (R, \beta\big|_S) \perp (S, \beta\big|_R)$ where R is an orthogonal direct sum of orthogonal hyperbolic lines and S is a one-dimensional unitary space. Then $L\beta\big|_R$ is of type O^+ by the first part of this lemma, $L\beta\big|_S$ is of type O^- by the previous argument and hence $(V, L\beta)$ is of type O^-.

Finally suppose that $w > 2$, in which case $L = Tr_{GF(q^{\frac{w}{2}})/GF(q)} \circ Tr_{GF(q^w)/GF(q^{\frac{w}{2}})}(\alpha)$. We know that $Tr_{GF(q^w)/GF(q^{\frac{w}{2}})}(\alpha\beta)$ is of type O^-; then Lemma 9 implies that $(V, L\beta)$ is of type O^-. $\square$

We are now in a position to summarise the results of the last two sections.

Theorem C. *Let V be an A-dimensional polar space over $GF(q^w)$. Take $L : GF(q^w) \to GF(q), x \mapsto Tr_{GF(q^w)/GF(q)}(\alpha x)$ for some $\alpha \in GF(q^w)^*$.*

Suppose first of all that V is defined via a quadratic form $Q : V \to GF(q^w)$. If the form has a germ U then $Q\big|_U(x) = \gamma x^2$ for some $\gamma \in GF(q^w)^*$. Then we classify LQ into type, including the classical group embedding,as follows:

Type of Q	*Type of LQ*	*Conditions*	*Embedding*
O^+	O^+	*always*	$O^+(A, q^w) \leq O^+(Aw, q)$
O^-	O^-	*always*	$O^-(A, q^w) \leq O^-(Aw, q)$
O	*degenerate*	*q even*	-
O	O	*w odd, q odd*	$O(A, q^w) \leq O(Aw, q)$
O	O^+	*w even, q odd;* $(\alpha\gamma)^{-2} \in GF(q^{\frac{w}{2}})\backslash GF(q^{\frac{w}{2}})^{*2}$ *or* $(\alpha\gamma)^{q+1} \not\equiv -1(mod\ GF(q^{\frac{w}{2}})^{*2})$	$O(A, q^w) \leq O^+(Aw, q)$
O	O^-	*w even, q odd;* $(\alpha\gamma)^{-2} \notin GF(q^{\frac{w}{2}})\backslash GF(q^{\frac{w}{2}})^{*2}$ *and* $(\alpha\gamma)^{q+1} \equiv -1(mod\ GF(q^{\frac{w}{2}})^{*2})$	$O(A, q^w) \leq O^-(Aw, q)$

Suppose next that V is defined via a reflexive σ-sesquilinear form $\beta : V \times V \to GF(q^w)$. If the characteristic is odd and β is symmetric then the type of $L\beta$ and its associated classical group embedding is given in the previous table taking Q to be the quadratic form $Q(v) = \frac{1}{2}\beta(v, v)$.

In all other cases the type of $L\beta$, with associated classical group embedding, is as follows:

Type of β	*Type of* $L\beta$	*Conditions*	*Embedding*
hermitian	*hermitian*	w *odd*, $\sigma(\alpha) = \alpha$	$U(A, q^w) \leq U(Aw, q)$
hermitian	*atypical*	w *odd*, $\sigma(\alpha) \neq \alpha$	-
hermitian	*alternating*	w *even*, q *even*, $\sigma(\alpha) = \alpha$	$U(A, q^w) \leq Sp(Aw, q)$
hermitian	*alternating*	w *even*, q *odd*, $\sigma(\alpha) = -\alpha$	$U(A, q^w) \leq Sp(Aw, q)$
hermitian	*atypical*	w *even*, $\sigma(\alpha) \neq \pm\alpha$	-
hermitian	O^+	w *even*, q *odd*, A *even*, $\sigma(\alpha) = \alpha$	$U(A, q^w) \leq O^+(Aw, q)$
hermitian	O^-	w *even*, q *odd*, A *odd*, $\sigma(\alpha) = \alpha$	$U(A, q^w) \leq O^-(Aw, q)$
alternating	*alternating*	*always*	$Sp(A, q^w) \leq Sp(Aw, q)$
pseudo -symplectic	*pseudo -symplectic*	q *even*	-

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